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Space-time discontinuous Galerkin Methods for the linear transport equation

Christian Wieners

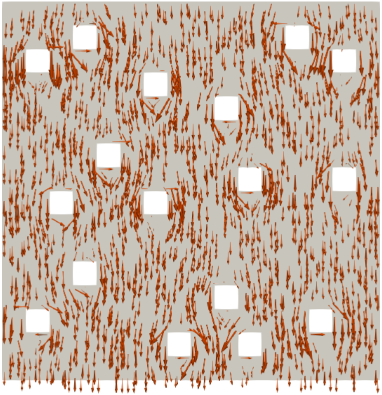
Institut für Angewandte und Numerische Mathematik

Application: Transport in porous media

Let $\Omega \subset (0,1)^2$ be a simplified configuration intersecting the top earth layers of sand with different permeability $\kappa \colon \overline{\Omega} \longrightarrow (\kappa_{\text{min}}, \kappa_{\text{max}}) \subset (0,\infty)$.

In this configuration $(0,1)^2 \setminus \Omega$ are impermeable stones and rocks.

Let $\Gamma_{\text{top}} = (0,1) \times \{1\} \subset \partial\Omega$ be the surface where it is raining,
and let $\Gamma_{\text{bottom}} = [0,1] \times \{0\} \subset \partial\Omega$ be the groundwater level.



First step

Compute the flux vector

$$\mathbf{q} \colon \overline{\Omega} \longrightarrow \mathbb{R}^2.$$

Second step

Starting with a pollution density

$$u^0 \colon \Omega \longrightarrow \mathbb{R}$$

compute the transport along $\mathbf{q}$

$$u \colon (0,T) \times \Omega \longrightarrow \mathbb{R}.$$

Application: Transport in porous media

The *porous media equation* is a Poisson problem for the *pressure head* $p\colon \Omega \longrightarrow \mathbb{R}$.

- $p(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma_{\text{bottom}}$ is fixed on the Dirichlet boundary $\Gamma_{\text{D}} = \Gamma_{\text{bottom}}$
- $\mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x}) = -1 \quad$ for $\mathbf{x} \in \Gamma_{\text{top}}$ with $\mathbf{n} = (0,1)^\top$ are the Neumann data
- $\mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x}) = \quad 0 \quad$ for $\mathbf{x} \in \partial\Omega \setminus (\Gamma_{\text{bottom}} \cap \Gamma_{\text{top}})$ with outer normal $\mathbf{n}$ defined a.e.
- $\mathbf{q}(\mathbf{x}) = -\boldsymbol{\kappa}(\mathbf{x})\nabla p(\mathbf{x})$ for $\mathbf{x} \in \Omega$ is the *material law*.

In weak form we have

$$\begin{aligned}\int_\Omega \boldsymbol{\kappa}(\mathbf{x})^{-1}\mathbf{q}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})\,\mathrm{d}\mathbf{x} &= -\int_\Omega \nabla p(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= -\int_{\Gamma_{\text{D}}} p_{\text{D}}(\mathbf{x})\mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})\,\mathrm{d}\mathbf{a} + \int_\Omega p(\mathbf{x})\nabla \cdot \boldsymbol{\phi}(\mathbf{x})\,\mathrm{d}\mathbf{x}\end{aligned}$$

for all smooth test function $\boldsymbol{\phi} \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^2)$ with $\mathbf{n} \cdot \boldsymbol{\phi} = 0$ on $\Gamma_{\text{N}} = \partial\Omega \setminus \Gamma_{\text{D}}$.
For all convex subset $K \subset \Omega$ we have the *balance law*

$$\int_{\partial K} \mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int_{\partial K \cap \Gamma_{\text{N}}} q_{\text{N}}(\mathbf{x})\,\mathrm{d}\mathbf{a}\,.$$

This implies for all test function $\psi \in \mathrm{C}^1(\overline{\Omega})$ with $\psi = 0$ on Γ_{D} or $\psi \in \mathrm{C}^1_{\text{c}}(\Omega \cup \Gamma_{\text{N}})$

$$\int_\Omega \operatorname{div}\mathbf{q}(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{q}(\mathbf{x}) \cdot \nabla\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Gamma_{\text{N}}} q_{\text{N}}(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mathbf{a}\,.$$

We have $p \in \mathrm{H}^1(\Omega)$ with $p = 0$ on Γ_{D}, $-\nabla \cdot \boldsymbol{\kappa}\nabla p = 0$ in Ω, and $\mathbf{q} \in \mathrm{H}(\operatorname{div},\Omega)$.

Application: Transport in porous media

For given *flux vector* $\mathbf{q}\colon \overline{\Omega} \longrightarrow \mathbb{R}^2$ and initial *pollution density* $u^0\colon \Omega \longrightarrow \mathbb{R}$ we compute the transport along $\mathbf{q}$ for the pollution density $u\colon (0,T) \times \Omega \longrightarrow \mathbb{R}$.

Here we assume no pollution inflow on $\Gamma_{\text{in}} = \{\mathbf{x} \in \partial\Omega\colon \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$.
The *conservation property* for all convex subsets $K \subset \Omega$ and $(t_1, t_2) \subset (0,T)$

$$\int_K \big(u(t_2,\mathbf{x}) - u(t_1,\mathbf{x})\big)\,\mathrm{d}\mathbf{x} + \int_{t_1}^{t_2} \int_{\partial K} u(t,\mathbf{x})\,\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t = 0$$

yields, in case that u and $\mathbf{q}$ are sufficiently smooth,

$$\int_{t_1}^{t_2} \int_K \big(\partial_t u(t,\mathbf{x}) + \operatorname{div}\big(u(t,\mathbf{x})\,\mathbf{q}(\mathbf{x})\big)\big)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} = 0\,.$$

This implies for all smooth test function $v \in \mathrm{C}^1_{\mathrm{c}}\big([0,T) \times (\Omega \cup \Gamma_{\text{in}})\big)$

$$0 = \int_0^T \int_\Omega u(t,\mathbf{x})\Big(- \partial_t v(t,\mathbf{x}) - \mathbf{q}(\mathbf{x}) \cdot \nabla v(t,\mathbf{x})\Big)\,\mathrm{d}(t,\mathbf{x})$$

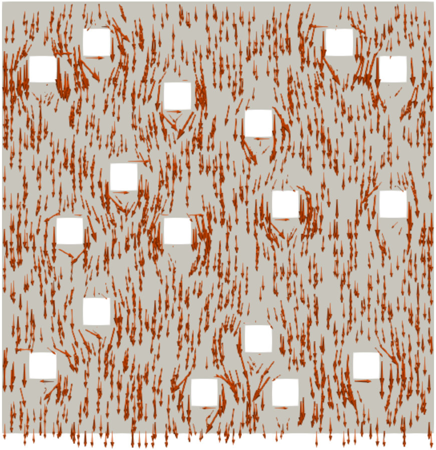
$$+ \int_\Omega \big(u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})\big)\,\mathrm{d}\mathbf{x} + \int_0^T \int_{\partial\Omega} u(t,\mathbf{x})\,\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})v(t,\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t$$

and thus (in case of no pollution inflow)

$$\int_0^T \int_\Omega u(t,\mathbf{x})\Big(\partial_t v(t,\mathbf{x}) + \mathbf{q}(\mathbf{x}) \cdot \nabla v(t,\mathbf{x})\Big)\,\mathrm{d}(t,\mathbf{x}) = \int_\Omega u^0(\mathbf{x})v(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Application: Transport in porous media

For given permeability $\boldsymbol{\kappa}$, solve $\operatorname{div} \mathbf{q} = 0$ with $\mathbf{q} = -\boldsymbol{\kappa}\nabla p$ in Ω and $p = 0$ on Γ_{D} .



Then, for given initial pollution density u^0, compute the transport with flux vector $\mathbf{q}$.
The solution with $23\,494\,656 = 128 \cdot 183\,552$ space-time DOFs
requires 2 minutes on 8 parallel cores and 40 seconds on 32 cores.

Approximation of the porous media equation

Let $\Omega \subset \mathbb{R}^d$ be a polygonal Lipschitz domain. For $h \in \mathcal{H} \subset (0, h_0)$ let $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ be meshes where the elements $K \subset \Omega$, $K \in \mathcal{K}_h$ are open triangles/tetrahedra.
Let $F \in \mathcal{F}_K$ be the faces of the element K, and we set $\mathcal{F}_h = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K$, so that $\partial\Omega_h = \overline{\bigcup_{F \in \mathcal{F}_h} F}$ is the skeleton in space, $\overline{\Omega} = \Omega_h \cup \partial\Omega_h$, and $\Gamma_{\mathsf{D}} = \overline{\bigcup_{F \in \mathcal{F}_h \cap \Gamma_{\mathsf{in}}} F}$.
We use Raviart-Thomas finite elements. Let

$$W_h = \Big\{ (p_h, \mathbf{q}_h) \in \mathrm{L}_2(\Omega) \times \mathrm{H}^1(\mathrm{div}, \Omega) \colon p_h|_K \in \mathrm{P}_0(K) \text{ for all } K \in \mathcal{K}_h \text{ and } \mathbf{q}_h|_K \in \mathrm{P}_1(K)^d \text{ such that } \mathbf{n}_F \cdot \mathbf{q}_h|_F \in \mathrm{P}_0(F) \text{ for all } F \in \mathcal{F}_K \Big\},$$

and compute $(p_h, \mathbf{q}_h) \in W_h$ with $\mathbf{n} \cdot \mathbf{q}_h = g_{\mathsf{N}}$ on Γ_{N} solving

$$\int_\Omega \Big(\boldsymbol{\kappa}^{-1} \mathbf{q}_h \cdot \boldsymbol{\phi}_h - p_h \,\mathrm{div}\, \boldsymbol{\phi}_h - \mathrm{div}\, \mathbf{q}_h \psi_h \Big) \,\mathrm{d}\mathbf{x} = - \int_{\Gamma_{\mathsf{D}}} p_{\mathsf{D}} \mathbf{n} \cdot \boldsymbol{\phi} \,\mathrm{d}\mathbf{a}$$

for all $(\psi_h, \boldsymbol{\phi}_h) \in W_h$ with $\mathbf{n} \cdot \boldsymbol{\phi}_h = 0$ on Γ_{N}.
The discretization is inf-sup stable, the solution is uniformly bounded in L_2, so that a weakly converging subsequence $\big(\mathbf{q}_h\big)_{h \in \mathcal{H}_0}$ with $\mathcal{H}_0 \subset \mathcal{H}$ and $0 \in \overline{\mathcal{H}_0}$ exists with weak limit $\mathbf{q} \in \mathrm{L}_2(\Omega; \mathbb{R}^d)$ and $\lim_{h \in \mathcal{H}_0} (\mathbf{q}_h, \boldsymbol{\phi})_\Omega = (\mathbf{q}, \boldsymbol{\phi})_\Omega$ for all $\boldsymbol{\phi} \in \mathrm{L}_2(\Omega; \mathbb{R}^d)$.

Outline

- Define a weak solution u in L_2 for the linear transport equation with flux vector $\mathbf{q}$ in L_2.
- Define a discontinuous space-time discretization space V_h which includes piecewise constant approximations in space and time.
- Define a variational approximation in time.
- Evaluate the upwind flux by the exact solution for the transport equation for a constant flux vector $\mathbf{q}$.
- Define a *space-time discontinuous Galerkin discretization* with *full upwind*.
- Establish *inf-sup stability* with respect to a suitable *mesh-dependent DG norm*. Thus, unique discrete approximations u_h exists and $\{u_h\}_h$ is uniformly *stable*.
- Establish *consistency* of the discrete solutions. This yields together with stability *convergence* to the weak solution.
- Construct a residual-type error indicator.
- Investigate numerically the convergence for smooth/non-smooth solutions.

The linear transport equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in space with Lipschitz boundary,
$I = (0,T)$ a time interval, and $Q = (0,T) \times \Omega$ the space-time cylinder.

We aim to compute the transport of a quantity $u\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}$ along a given vector field $\mathbf{q}\colon \overline{\Omega} \longrightarrow \mathbb{R}^d$. The corresponding flux function $\mathbf{f}$ is given by $\mathbf{f}(u) = u\,\mathbf{q}$.

- conservation property for all convex subsets $K \subset \Omega$ and $(t_1, t_2) \subset (0,T)$

$$\int_K \big(u(t_2,\mathbf{x}) - u(t_1,\mathbf{x})\big)\,\mathrm{d}\mathbf{x} + \int_{t_1}^{t_2} \int_{\partial K} u(t,\mathbf{x})\,\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t = 0\,,$$

- initial condition and boundary condition on $\Gamma_{\text{in}} = \{\mathbf{x} \in \partial\Omega\colon \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$

$$u = u^0 \quad \text{for} \quad \{0\} \times \Omega\,, \qquad\qquad \mathbf{f}(u) \cdot \mathbf{n} = g_{\text{in}} \quad \text{for} \quad (0,T) \times \Gamma_{\text{in}}\,.$$

If u and $\mathbf{q}$ are sufficiently smooth, we have $\partial_t u + \operatorname{div} \mathbf{f}(u) = 0$ in $(0,T) \times \Omega$.

Definition

For $\mathbf{q} \in \mathrm{L}_2(\Omega;\mathbb{R}^d)$, $u^0 \in \mathrm{L}_2(\Omega)$, and $g_{\text{in}} \in \mathrm{L}_2((0,T) \times \Gamma_{\text{in}})$, $\Gamma_{\text{out}} = \partial\Omega \setminus \Gamma_{\text{in}}$,
a *weak solution* $u \in \mathrm{L}_2(Q)$ of the linear transport equation is defined by

$$\int_Q u\Big(-\partial_t v - \mathbf{q} \cdot \nabla v\Big)\,\mathrm{d}(t,\mathbf{x}) = \int_\Omega u^0 v(0)\,\mathrm{d}\mathbf{x} - \int_0^T \int_{\Gamma_{\text{in}}} g_{\text{in}} v\,\mathrm{d}\mathbf{a}\,\mathrm{d}t\,,$$

$$v \in \mathcal{V}^* := \big\{v \in \mathrm{C}^1(\overline{Q})\colon v = 0 \text{ on } \{T\} \times \Omega \cup (0,T) \times \Gamma_{\text{out}}\big\}\,.$$

The weak linear problem

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We construct a discretization for the linear problem to find $u \in \mathrm{L}_2(Q)$ solving

$$b(u,w) = \ell(w)\,, \qquad w \in \mathcal{V}^*$$

with

$$b(v,w) = m(v,w) + \int_0^T a(v(t), w(t))\,\mathrm{d}t$$

and

$$m(v,w) = -\int_Q v\partial_t w\,\mathrm{d}(t,\mathbf{x})\,,$$

$$a(v(t), w(t)) = -\int_\Omega \mathbf{f}(v(t)) \cdot \nabla w(t)\,\mathrm{d}\mathbf{x}\,,$$

$$\ell(w) = \int_\Omega u^0 w(0)\,\mathrm{d}\mathbf{x} - \int_0^T \int_{\Gamma_{\text{in}}} g_{\text{in}}(t) w(t)\,\mathrm{d}\mathbf{a}\,\mathrm{d}t$$

using the notation $v(t) = v(t,\cdot) \in \mathrm{L}_2(\Omega)$.

The DG finite element space in the space-time cylinder

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For $0 = t_0 < t_1 < \cdots < t_N = T$, we define time intervals $I_{n,h} = (t_{n-1}, t_n)$ and

$$I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)\,, \qquad \partial I_h = \{t_0, t_1, \ldots, t_{N-1}, t_N\}\,.$$

Let $\mathcal{K}_h$ be a mesh so that $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ is a decomposition in space into open cells $K \subset \Omega \subset \mathbb{R}^d$. We obtain a decomposition into $R = I_{n,h} \times K$ and

$$Q_h = I_h \times \Omega_h = \bigcup_{n=1}^{N} Q_{n,h} = \bigcup_{R \in \mathcal{R}_h} R\,, \qquad Q_{n,h} = \bigcup_{K \in \mathcal{K}_h} I_{n,h} \times K \;\subset\; I_{n,h} \times \Omega\,.$$

In order to calibrate the accuracy in space and time, we assume

$$c_{\text{ref}} \triangle t \le h\,, \qquad \triangle t = \max(t_n - t_{n-1})\,, \qquad h = \max \operatorname{diam}(K)\,,$$

where $c_{\text{ref}} > 0$ is a reference velocity depending on the flux vector $\mathbf{q}$.

The DG discretization in space and time is defined for $V_h \subset \mathcal{V}_h \subset \mathrm{H}^1(Q_h)$.
For $v_h \in \mathcal{V}_h$ define $v_{n,h} = v_h|_{Q_{h,n}} \in \mathrm{H}^1(Q_{h,n})$.
This implies $v_{n,h}(t_{n-1}) \in \mathrm{L}_2(\Omega)$ and $v_{n,h}(t_n) \in \mathrm{L}_2(\Omega)$,
but $v_{n,h}(t_n)$ and $v_{n+1,h}(t_n)$ may be different.

The DG discretization in space is defined for $S_{n,h} \subset \mathcal{S}_h \subset \mathrm{H}^1(\Omega_h)$.
For $v_h \in \mathcal{S}_h$ define $v_{n,K} = v_h|_K \in \mathrm{H}^1(K)$.
This implies $v_{n,K}|_F \in \mathrm{L}_2(F)$ for $F \in \mathcal{F}_K$,
but $v_{n,K}|_F$ and $v_{n,K'}|_F$ may be different for $F \in \mathcal{F}_K \cap \mathcal{F}_{K'}$.

Full upwind in time

For $v_h, w_h \in \mathcal{V}_h$ we obtain after integration by parts in all intervals $I_{n,h} \subset I_h$

$$\big(\partial_t v_h, w_h\big)_{Q_h} = \sum_{n=1}^{N} \Big(- \big(v_{n,h}, \partial_t w_{n,h}\big)_{Q_{n,h}} + \big(v_{n,h}(t_n), w_{n,h}(t_n)\big)_\Omega - \big(v_{n,h}(t_{n-1}), w_{n,h}(t_{n-1})\big)_\Omega \Big) .$$

With $[w_h]_n = w_{n+1,h}(t_n) - w_{n,h}(t_n)$, $n = 1, \dots, N-1$ and $[w_h]_N = -w_{N,h}(t_N)$ set

$$m_h(v_h, w_h) = \sum_{n=1}^{N} \Big(- \big(v_{n,h}, \partial_t w_{n,h}\big)_{Q_{n,h}} - \big(v_{n,h}(t_n), [w_h]_n\big)_\Omega \Big) .$$

Integrating by parts and defining $[v_h]_0 = v_{1,h}(0)$ yields

$$m_h(v_h, w_h) = \big(\partial_t v_h, w_h\big)_{Q_h} + \sum_{n=1}^{N} \big([v_h]_{n-1}, w_{n,h}(t_{n-1})\big)_\Omega .$$

Together, we obtain

$$m_h(v_h, v_h) = \frac{1}{2} \sum_{n=0}^{N} \big\| [v_h]_n \big\|_\Omega^2 \geq 0 .$$

For test functions $w \in \mathrm{H}^1\big(0, T; \mathrm{L}_2(\Omega)\big)$ with $w(T) = 0$ we get *consistency*, i.e.,

$$m_h(v_h, w) = -\big(v_h, \partial_t w\big)_{Q_h} = m(v_h, w) .$$

The Riemann problem

We consider the special case $\mathbf{q} \in \mathbb{R}^d$ and
constant initial values $u^-, u^+ \in \mathbb{R}$ for $\mathbf{x} \cdot \mathbf{n} < 0$ and $\mathbf{x} \cdot \mathbf{n} > 0$ with $\mathbf{n} \in \mathbb{R}^d$, $\mathbf{n} \cdot \mathbf{n} = 1$.
Then we define the piecewise constant function $u \in \mathrm{L}_2(Q)$ by

$$u(t, \mathbf{x}) = \begin{cases} u^- \,, & (\mathbf{x} - t\mathbf{q}) \cdot \mathbf{n} < 0 \,, \\ u^+ \,, & (\mathbf{x} - t\mathbf{q}) \cdot \mathbf{n} > 0 \,. \end{cases}$$

We observe for $v \in \mathrm{C}^1_\mathrm{c}(Q)$

$$\begin{aligned} \big(u, -\partial_t v - \mathbf{q} \cdot \nabla v\big)_Q &= \int_Q u(t,\mathbf{x}) \begin{pmatrix} 1 \\ \mathbf{q} \end{pmatrix} \cdot \begin{pmatrix} -\partial_t v(t,\mathbf{x}) \\ -\nabla v(t,\mathbf{x}) \end{pmatrix} \mathrm{d}(t,\mathbf{x}) \\ &= \int_{\{(t,\mathbf{x}) \in Q \colon (\mathbf{x} - t\mathbf{q}) \cdot \mathbf{n} = 0\}} (u^+ - u^-) v(t,\mathbf{x}) \begin{pmatrix} 1 \\ \mathbf{q} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{q} \cdot \mathbf{n} \\ -\mathbf{n} \end{pmatrix} \mathrm{d}\mathbf{a}(t,\mathbf{x}) = 0 \,, \end{aligned}$$

i.e., u is a weak solution of the transport equation.

For $t > 0$ and $\mathbf{x} \cdot \mathbf{n} = 0$ we obtain

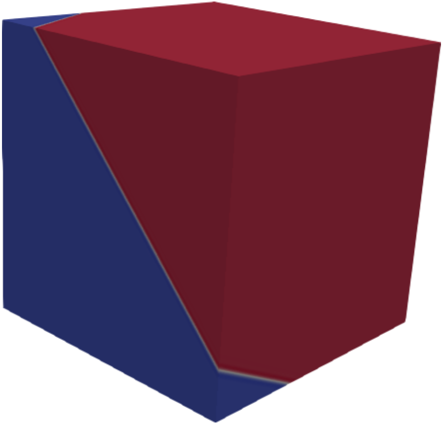
$$\begin{aligned} u(t,\mathbf{x}) &= u^- \quad \text{if} \quad \mathbf{q} \cdot \mathbf{n} > 0 \,, \\ u(t,\mathbf{x}) &= u^+ \quad \text{if} \quad \mathbf{q} \cdot \mathbf{n} < 0 \,. \end{aligned}$$

This now defines the upwind flux.

Example for the Riemann problem

The discontinuous piecewise constant solution of the Riemann problem:

$$u^0(\mathbf{x}) = \begin{cases} 1\,, & x_1 + 2x_2 + 5/9 < 0 \\ 0\,, & \text{else} \end{cases} \qquad \mathbf{q} = \begin{pmatrix} 2 \\ 0.5 \end{pmatrix}$$



Full upwind in space

For $v_h, w_h \in \mathcal{S}_h$ we observe for the discrete flux $\mathbf{f}_h(v_h) = v_h\, \mathbf{q}_h$

$$\big(\operatorname{div} \mathbf{f}_h(v_h), w_h\big)_{\Omega_h} = \sum_{K \in \mathcal{K}_h} \Big(-\big(\mathbf{f}_h(v_{h,K}), \nabla w_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\mathbf{f}_h(v_{h,K}) \cdot \mathbf{n}_K, w_{h,K}\big)_F \Big).$$

For discontinuous functions $v_h \in V_h$, this is approximated by the upwind flux

$$a_h(v_h, w_h) = \sum_{K \in \mathcal{K}_h} \Big(-\big(\mathbf{f}_h(v_{h,K}), \nabla w_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\mathbf{f}^{\text{up}}_{K,F}(v_h) \cdot \mathbf{n}_K, w_{h,K}\big)_F \Big),$$

$$\mathbf{f}^{\text{up}}_{K,F}(v_h) = \begin{cases} \mathbf{f}_h(v_{h,K}), & F \in \mathcal{F}_K^{\text{out}} \\ \mathbf{f}_h(v_{h,K_F}), & F \in \mathcal{F}_K^{\text{in}} \setminus \Gamma_{\text{in}} \\ \mathbf{0}, & F \in \mathcal{F}_K^{\text{in}} \cap \Gamma_{\text{in}} \end{cases} \quad \text{with} \quad \begin{cases} \mathcal{F}_K^{\text{out}} = \{F \in \mathcal{F}_K : \mathbf{q}_h \cdot \mathbf{n}_K \geq 0 \text{ on } F\} \\ \mathcal{F}_K^{\text{in}} = \{F \in \mathcal{F}_K : \mathbf{q}_h \cdot \mathbf{n}_K < 0 \text{ on } F\} \end{cases}$$

with $\overline{F} = \partial K \cap K_F$ and assuming that $\mathbf{q}_h \cdot \mathbf{n}_K$ is constant on F.

Defining $[v_h]_{K,F} = v_{h,K_F} - v_{h,K}$ on inner faces $F \in \mathcal{F}_h \cap \Omega$, we obtain

Lemma

$$a_h(v_h, v_h) = \frac{1}{2} \int_{\Omega_h} v_h^2 \operatorname{div} \mathbf{q}_h \, \mathrm{d}\mathbf{x} + \frac{1}{4} \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K \cap \Omega} \int_F \big([v_h]_{K,F}\big)^2 |\mathbf{q}_h \cdot \mathbf{n}_K| \, \mathrm{d}\mathbf{a}$$
$$+ \frac{1}{2} \int_{\partial\Omega} v_h^2 \, |\mathbf{q}_h \cdot \mathbf{n}| \, \mathrm{d}\mathbf{a}, \qquad v_h \in \mathcal{S}_h .$$

The full upwind method in space and time

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The discrete bilinear form is defined by

$$b_h(v_h, w_h) = m_h(v_h, w_h) + \int_0^T a_h(v_h(t), w_h(t))\, \mathrm{d}t\,, \qquad v_h, w_h \in \mathcal{V}_h\,.$$

We have consistency up to the data error

$$b_h(v_h, w) = b(v_h, w) + \int_Q v_h(\mathbf{q} - \mathbf{q}_h) \cdot \nabla w \, \mathrm{d}(t, \mathbf{x})\,, \qquad v_h \in \mathcal{V}_h,\ w \in \mathcal{V}^*$$

and

$$b_h(v_h, v_h) = \frac{1}{2} \sum_{n=0}^{N} \big\| [v_h]_n \big\|_\Omega^2 + \frac{1}{2} \big(v_h \operatorname{div} \mathbf{q}_h, v_h \big)_Q + \frac{1}{2} \sum_{F \in \mathcal{F}_h \cap \Omega} \Big\| |\mathbf{q}_h \cdot \mathbf{n}_K|^{1/2} [v_h]_{K,F} \Big\|_{I \times F}^2 + \frac{1}{2} \Big\| |\mathbf{q}_h \cdot \mathbf{n}|^{1/2} v_h \Big\|_{I \times \partial\Omega}^2 .$$

Lemma

Define $d_T(t) = T - t$. *If* $\operatorname{div} \mathbf{q}_h \geq 0$, *we have*

$$\|v_h\|_{Q_h}^2 + T\, \|v_h(0)\|_{\Omega_h}^2 \leq 2\, b_h(v_h, d_T v_h)\,, \qquad v_h \in \mathcal{V}_h\,.$$

Well-posedness of the DG method

We assume for the approximation of the flux vector

A1) $\operatorname{div} \mathbf{q}_h \geq 0$,

A2) $\operatorname{div}(v_h \mathbf{q}_h) \in V_h$ for all $v_h \in V_h$,

A3) inflow and outflow boundary characterized from $\mathbf{q}$ and $\mathbf{q}_h$ coincide:

$$\overline{\Gamma}_{\text{in}} = \bigcup_{F \in \mathcal{F}_h^{\text{in}}} \overline{F}\,, \quad \overline{\Gamma}_{\text{out}} = \bigcup_{F \in \mathcal{F}_h^{\text{out}}} \overline{F} \quad \text{with} \quad \mathcal{F}_h^{\text{in}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{in}}\,, \quad \mathcal{F}_h^{\text{out}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{out}}\,.$$

Lemma

A unique Galerkin approximation $u_h \in V_h$ exists solving

$$b_h(u_h, v_h) = \ell(v_h)\,, \qquad v_h \in V_h\,.$$

Finite element spaces

Set $h_K = \operatorname{diam} K$, $h_F = \operatorname{diam} F$, $h = \max h_K$, assume $h_F \geq c_{\mathsf{sr}} h_K$, $h_K \geq c_{\mathsf{mr}} h$.

Select polynomial degrees $p_R = p_{n,K} \geq 0$, $s_R = s_{n,K} \geq 0$ in time and space, set

$$S_{n,h} = \prod_{K \in \mathcal{K}_h} \mathbb{P}_{s_{n,K}}(K)\,, S_h = \sum_{n=1}^{N} S_{n,h}\,, V_{n,h} = \prod_{K \in \mathcal{K}_h} \mathbb{P}_{p_{n,K}} \otimes \mathbb{P}_{s_{n,K}}(K)\,, V_h = \sum_{n=1}^{N} V_{n,h}\,.$$

For a sequence of mesh sizes $\mathcal{H} = \{h_0, h_1, h_2, \cdots\} \subset (0, \infty)$ and $0 \in \overline{\mathcal{H}}$,
let $\big(Q_h\big)_{h \in \mathcal{H}}$ be a shape-regular family of space-time meshes and
let $\big(V_h\big)_{h \in \mathcal{H}}$ the corresponding DG finite element spaces, so that

$$\lim_{h \in \mathcal{H}} \inf_{v_h \in V_h} \big\| v - v_h \big\|_Q = 0\,, \qquad v \in \mathrm{L}_2(Q)\,.$$

Depending on the space-time mesh regularity and on $\mathbf{q}_h$, $C_{\mathsf{inv}}, C_{\mathsf{tr}} > 0$ exists with

$$\big\| h^{1/2} \left(\partial_t v_h + \operatorname{div}(v_h \mathbf{q}_h)\right) \big\|_{Q_h} \leq C_{\mathsf{inv}} \big\| h^{-1/2} v_h \big\|_{Q_h}\,, \qquad \big\| v_h \big\|_{\partial Q_h} \leq C_{\mathsf{tr}} \big\| h^{-1/2} v_h \big\|_{Q_h}\,.$$

If $v_h \in \mathrm{H}^1(Q)$, a stable quasi-interpolation $v_h = \Pi_h^{\mathsf{Cl}} u$ be of Clement-type exists with

$$h^{-1} \big\| u - \Pi_h^{\mathsf{Cl}} u \big\|_Q + \big\| \partial_t \big(u - \Pi_h^{\mathsf{Cl}} u\big) \big\|_Q + \big\| \operatorname{div} \mathbf{f}(u) - \operatorname{div} \mathbf{f}(\Pi_h^{\mathsf{Cl}} u) \big\|_Q \leq C_{\mathsf{Cl}} \big\| \mathrm{D} u \big\|_Q$$

and a constant $C_{\mathsf{Cl}} > 0$ depending on the mesh and the polynomial degrees.

Inf-sup stability of the full-upwind method

For all $v_h \in \mathcal{V}_h$ we define the DG semi-norm and norm

$$\begin{aligned}
\big|v_h\big|^2_{h,\mathrm{DG}} =& \frac{1}{2}\sum_{n=0}^{N} \big\|[v_h]_n\big\|^2_{\Omega} \\
&+ \frac{1}{4}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega} \big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[v_h]_{K,F}\big\|^2_{I_h\times\partial K} + \frac{1}{2}\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2} v_h\big\|^2_{I_h\times\partial\Omega}\,, \\
\big\|v_h\big\|_{h,\mathrm{DG}} =& \sqrt{\big|v_h\big|^2_{h,\mathrm{DG}} + \big\|h^{1/2}\left(\partial_t v_h + \mathrm{div}(v_h\mathbf{q}_h)\right)\big\|^2_{Q_h}}\,.
\end{aligned}$$

Theorem

A constant $c_{\text{inf-sup}} > 0$ independent of the mesh size h exists such that

$$\sup_{w_h\in V_h\setminus\{0\}} \frac{b_h(v_h,w_h)}{\big\|w_h\big\|_{h,\mathrm{DG}}} \geq c_{\text{inf-sup}}\big\|v_h\big\|_{h,\mathrm{DG}}\,, \qquad v_h \in V_h\,.$$

Convergence of the DG space-time approximation

The space-time trace estimate

$$\frac{1}{2}\left\|v_h(0)\right\|_\Omega^2 + \frac{1}{2}\left\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}v_h\right\|_{I_h\times\Gamma_{\text{in}}}^2 \le \left|v_h\right|_{h,\mathrm{DG}}^2\,, \qquad v_h\in V_h\,,$$

and inf-sup stability implies for the discrete solution $u_h \in V_h$

$$c_{\text{inf-sup}}\left\|u_h\right\|_{h,\mathrm{DG}} \le 2\left\|u^0\right\|_\Omega + 2\left\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}g_{\text{in}}\right\|_{I_h\times\Gamma_{\text{in}}}\,.$$

Lemma

Assume for the approximation of the flux vector

1) $C_{\text{in}} > 0$ *exists s.t.* $\left\|\,|\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}g_{\text{in}}\right\|_{I\times\Gamma_{in}} \le C_{\text{in}}$ *is uniformly bounded for* $h\in\mathcal{H}$*;*
2) $C_{\mathbf{q}} > 0$ *exists such that* $\left\|\,|\mathbf{q}_h\cdot\mathbf{n}|^{1/2}u_h\right\|_{I_h\times\partial\Omega_h} \le C_{\mathbf{q}}\left\|u_h\right\|_{I_h\times\partial\Omega_h}$ *for all* $h\in\mathcal{H}$*;*
3) *strong convergence in* L_2*, i.e.,* $\lim_{h\in\mathcal{H}}\left\|\mathbf{q}_h-\mathbf{q}\right\|_\Omega = 0$.

Then,

a) $(u_h)_{h\in\mathcal{H}}$ *is weakly converging in* $\mathrm{L}_2(Q)$*;*
b) *the weak limit* $u\in\mathrm{L}_2(Q)$ *is a weak solution;*
c) *the weak solution* $u\in\mathrm{L}_2(Q)$ *is unique;*
d) *the weak solution is also a strong solution and* $\partial_t u + \operatorname{div}\mathbf{f}(u)\in\mathrm{L}_2(Q)$.

Qualitative convergence estimates in the DG norm

KIT Karlsruhe Institute of Technology

Theorem

Assume that the solution u is sufficiently smooth satisfying $u \in \mathrm{H}^r(Q)$ with

$$1 \le r \le \min_{n,K}\{p_{n,K}, s_{n,K}\} + 1 \,.$$

Then, the error for the discrete solution $u_h \in V_h$ is bounded by

$$\big\|u - u_h\big\|_{h,\mathrm{DG}} \le C_1 h^{r-1/2} \big\|\mathrm{D}^r u\big\|_Q + C_2 T h^{-1/2} \big\| \operatorname{div} \mathbf{f}(u) - \operatorname{div} \mathbf{f}_h(u)\big\|_Q$$

with $C_1, C_2 > 0$ depending on mesh regularity, polynomial degrees in V_h, and $\mathbf{q}$.

$\big\|u(t_n) - u_h(t_n)\big\|_\Omega \le t_n \big\|u - u_h\big\|_{h,\mathrm{DG}} = \mathcal{O}(h^{r-1/2})$ is optimal for $u(t_n) \in \mathrm{H}^{r-1/2}(Q)$.

If the flux vector $\mathbf{q}$ is sufficiently smooth, the consistency term can be estimated by

$$\big\| \operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\big\|_Q \le \big\| \operatorname{div} \mathbf{q} - \operatorname{div} \mathbf{q}_h\big\|_\infty \big\|u\big\|_Q + \big\|\mathbf{q} - \mathbf{q}_h\big\|_\infty \big\|\nabla u\big\|_Q \,.$$

In general only $\mathbf{q} \in \mathrm{H}(\operatorname{div}, \Omega)$ can be assumed. If $u \in \mathrm{L}_2(0,T;\mathrm{W}^1_r(\Omega))$ with $r > 2$

$$\begin{aligned}\big\| \operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\big\|_Q &\le \big\| \operatorname{div}(\mathbf{q} - \mathbf{q}_h)\big\|_\Omega \big\|u\big\|_Q \\ &\quad + \big\|\mathbf{q} - \mathbf{q}_h\big\|_{\mathrm{L}_{r/(r-1)}(\Omega;\mathbb{R}^d)} \big\|\nabla u\big\|_{\mathrm{L}_2(0,T;\mathrm{L}_r(\Omega;\mathbb{R}^d))} \,.\end{aligned}$$

Error control

The error $u - u_h$ in the DG semi-norm takes the form

$$\big|u - u_h\big|_{h,\mathrm{DG}} = \Bigg(\frac{1}{2}\big\|u^0 - u_h(0)\big\|_\Omega^2 + \frac{1}{2}\sum_{n=1}^{N-1} \big\|[u_h]_n\big\|_\Omega^2 + \frac{1}{2}\big\|u(T) - u_h(T)\big\|_\Omega^2$$

$$+ \frac{1}{4}\sum_{K\in\mathcal{K}_h}\ \sum_{F\in\mathcal{F}_K\cap\Omega} \big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|_{I_h\times\partial K}^2 + \frac{1}{2}\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u - u_h)\big\|_{I_h\times(\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}})}^2 \Bigg)^{1/2}$$

and for the DG norm we get

$$\big\|u - u_h\big\|_{h,\mathrm{DG}}^2 \le \big|u - u_h\big|_{h,\mathrm{DG}}^2 + 2\,\big\|h^{1/2}\left(\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h)\right)\big\|_{Q_h}^2$$
$$+ 2\,\big\|h^{1/2}\ \operatorname{div}(\mathbf{f}(u) - \mathbf{f}_h(u))\big\|_{Q_h}^2 .$$

Up to the error $u_h - u$ at final time T and on the outflow boundary
and without estimating the consistency error
this is explicitly evaluated by the residual error indicator

$$\eta_{\mathrm{res},h} = \Bigg(\sum_{R\in\mathcal{R}_h} \eta_{\mathrm{res},R}^2\Bigg)^{1/2}$$

Error control

$\eta_{\text{res},h}$ is given by the local contributions for $R = (t_{n-1}, t_n) \times K$, $n = 1, \dots, N$

$$\eta^2_{\text{res},R} = \eta^2_{\text{res},n,K} + 2h_K \left\| \partial_t u_h + \operatorname{div} \mathbf{f}_h(u_h) \right\|^2_R$$
$$+ \frac{1}{4} \left\| |\mathbf{q}_h \cdot \mathbf{n}_K|^{1/2} [u_h]_{K,F} \right\|^2_{I_h \times \partial K \cap \Omega} + \left\| |\mathbf{q}_h \cdot \mathbf{n}|^{-1/2} (g_{\text{in}} - \mathbf{f}_h(u_h) \cdot \mathbf{n}) \right\|^2_{I_h \times \partial K \cap \Gamma_{\text{in}}}$$

$$\eta^2_{\text{res},1,K} = \frac{1}{2} \left\| u^0 - u_h(0) \right\|^2_K + \frac{1}{4} \left\| [u_h]_1 \right\|^2_K , \qquad R = (0, t_1) \times K ,$$

$$\eta^2_{\text{res},n,K} = \frac{1}{4} \left\| [u_h]_{n-1} \right\|^2_K + \frac{1}{4} \left\| [u_h]_n \right\|^2_K , \qquad R = (t_{n-1}, t_n) \times K ,\ 1 < n < N ,$$

$$\eta^2_{\text{res},N,K} = \frac{1}{4} \left\| [u_h]_{N-1} \right\|^2_K , \qquad R = (t_{N-1}, T) \times K .$$

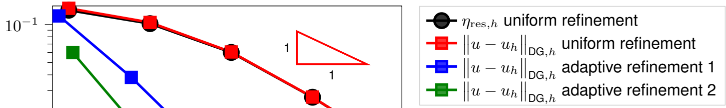
Lemma

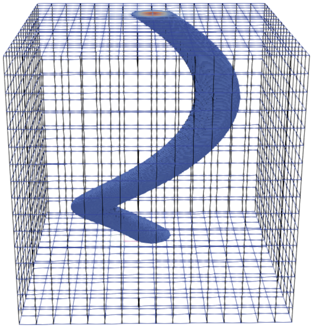
Let $u \in \mathrm{L}_2(Q)$ be the weak solution and $u_h \in V_h$ the discrete solution. Then, if u is sufficiently smooth, the error in the DG norm is bounded by

$$\left\| u - u_h \right\|_{h,\text{DG}} \le \Big(\eta^2_{\text{res},h} + \frac{1}{2} \left\| (u_h(T) - u(T)) \right\|^2_\Omega + \frac{1}{2} \left\| |\mathbf{q}_h \cdot \mathbf{n}|^{1/2} (u - u_h) \right\|^2_{I_h \times \Gamma_{\text{out}}}$$
$$+ 2 \left\| h^{1/2} \operatorname{div}(\mathbf{f}(u) - \mathbf{f}_h(u)) \right\|^2_{Q_h} + \left\| |\mathbf{q}_h \cdot \mathbf{n}|^{-1/2} (\mathbf{f}(u) - \mathbf{f}_h(u)) \cdot \mathbf{n} \right\|^2_{I_h \times \Gamma_{\text{in}}} \Big)^{1/2} .$$

Example: Rotation cone

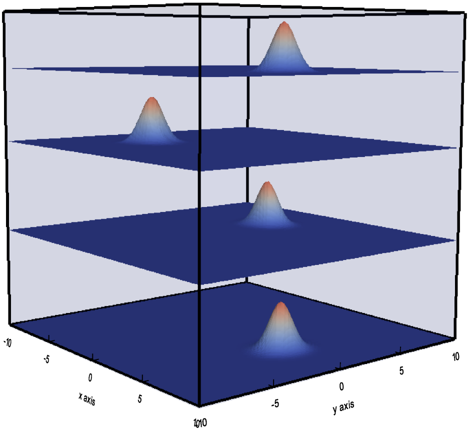
$$u(t,\mathbf{x}) = \begin{cases} \exp\left(-\dfrac{|\mathbf{x}-\mathbf{y}(t)|}{\frac{1}{8}-|\mathbf{x}-\mathbf{y}(t)|}\right), & |\mathbf{x}-\mathbf{y}(t)| < \frac{1}{8}, \quad \mathbf{y}(t) = \frac{1}{4}\begin{pmatrix}\cos(2\pi t)\\ \sin(2\pi t)\end{pmatrix}, \\ 0, & \text{else,} \qquad \mathbf{q}(\mathbf{x}) = \begin{pmatrix}-2\pi x_2\\ 2\pi x_1\end{pmatrix}. \end{cases}$$





Example: Rotation cone

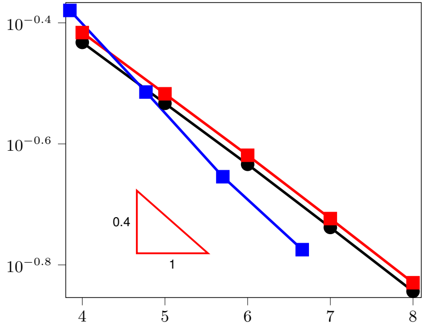
The adaptive solution with 3 303 810 degrees of freedom computes the same solution as the uniform computation on $524\,288 = 4\,096 \times 128$ space-time cells and 31 703 040 degrees of freedom.

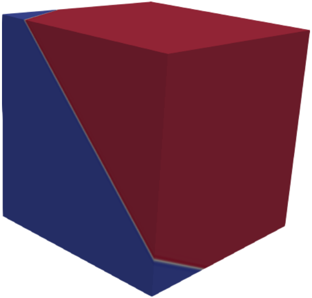


solution sliced at times $t = 0, 0.3, 0.6, 1$

Example: The Riemann problem

Since the Riemann solution is not in $\mathrm{H}^1(Q)$, the theorem does not apply.





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