On reconstruction based discontinuous Galerkin methods, an inverse L^2-projection, and entropy stable flux limiting

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Two of the most popular families of high order schemes for hyperbolic conservation laws are *reconstruction-based finite volume* schemes and *discontinuous Galerkin (DG)* methods. The $P_N P_M$ philosophy presented in [1] provides a unified framework for the treatment of both approaches. In a $P_N P_M$ scheme, the solution is represented in a finite element space of piecewise polynomials of degree $N \geq 0$ (hence the P_N in the name of the method) and at each time step before the time evolution is carried out, a high order reconstruction of piecewise polynomials of degree $M \geq N$ is computed (hence the P_M). In this framework, the pure DG method can be viewed as a $P_N P_N$ scheme, while the case $P_0 P_M$ corresponds to the high order finite volume schemes. For $N > 0$ and $M > N$ a family of hybrid schemes emerges.

We study some theoretical aspects of $P_N P_M$ schemes and show analytically why these methods are, in general, not L^2-diminishing. To this end, we extend the famous cell square entropy stability result of Jiang and Shu [2] for DG methods to the $P_N P_M$ case and identify which part in the reconstruction step may cause the instability. With this insight we design a flux limiter that enforces a cell square entropy condition for $P_N P_M$ schemes in $1D$.

Moreover, we provide a proof that the reconstruction method that is usually applied in $P_N P_M$ methods, which can be interpreted as an inverse L^2-projection, has a unique solution for every $N \geq 0$. It seems like such a result for arbitrary N has not been available in the literature so far.

References

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